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Rings of homogeneous functions

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Abstract

We investigate the question "When is the near-ring, $M_R(V)$, of homogeneous functions a ring?" for modules over commutative Noetherian rings. Particular attention is directed to the case in which V is an injective module. © 1998 Elsevier Science B.V.

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1. Introduction

Let R be a ring with identity and V a unital R-module. The set, $M_R(V) = \{f : V \to V \mid f(rv) = rf(v),\ \forall r \in R,\ \forall v \in V\}$, with the operations of function addition and function composition is a zero-symmetric near-ring with identity called the *centralizer near-ring determined by* (R, V) or the *near-ring of homogeneous functions determined by* (R, V). This near-ring has been the object of several investigations. In [3], the problem of characterizing those rings R such that $M_R(V)$ is a ring for all R-modules V was initiated. This line of investigation was continued by Hausen and Johnson in [4], in which they characterized all those modules V over a Dedekind domain, D, for which $M_D(V)$ is a ring and also for which $M_D(V) = End_D(V)$. The structure of $M_R(V)$ was investigated in [2] for finitely generated modules V over principal ideal domains R.

In this work we turn our attention to arbitrary commutative Noetherian rings. Since every module can be embedded in an injective module, the following very nice structural result of Matlis [5] is very useful in our situation. (See [9] for an exposition of this result.)

Theorem ([5, 9]). *Let R be a commutative Noetherian ring.*

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(i) *Every injective R-module is uniquely a direct sum of uniform injective modules.*

(ii) *The map $P \mapsto E(R/P)$ yields a one–one correspondence between the prime ideals P of R and the isomorphism classes of uniform injective R-modules, where $E(R/P)$ is the injective hull of R/P.*

(iii) *If P is a prime ideal of R, then every element of $E(R/P)$ is annihilated by some power of P.*

From this result, if one focuses on uniform injective or injective modules much can be said. When investigating near-rings of homogeneous functions, a natural first question to ask is "When is $M_R(V)$ a near-field?". (See [6].) Using a result from [14], we give a complete answer (Theorem 2.2) to this question for modules over commutative rings.

Following the lead of Hausen and Johnson one might then ask "When is $M_R(V)$ a ring?" or "When is $M_R(V) = End_R(V)$?". We consider these questions in Section 2, finding in many situations that $M_R(V)$ is a ring if and only if it is a commutative ring. For finitely generated injective modules, we characterize when $M_R(V)$ is a ring.

To the best of the authors' knowledge, all previous examples of rings of homogeneous functions, $M_R(V)$, turned out to be commutative when R is a commutative ring. We explore this relationship further, finding that this is indeed the case in many instances but also showing that there exist modules V over a commutative ring R such that $M_R(V)$ is a ring but is not commutative.

In Section 3 we study a particular class of examples. Besides providing some examples of our results and some counter examples to possible variations, perhaps the main reason for considering this class of examples is to provide new examples of uniform injective modules with which one can calculate. Although there is extensive literature on injective hulls there are very few concrete examples other than the types provided by modules over Dedekind domains and modifications of these. Our examples should help fill this gap.

Conventions. Unless stated to the contrary, all rings will be Noetherian, commutative, and with identity. All modules will be unitary. We will denote the collection of R-modules by mod-R and the injective hull of an R-module V by $E(V)$ or $E_R(V)$ when we need to emphasize the ring.

2. When is $M_R(V)$ a ring?

The main purpose of this section is to investigate the above question. However, we first determine when $M_R(V)$ is a near-field for any commutative ring R and $V \in$ mod -R. When $M_R(V)$ is a near-field, then each nonzero $f \in M_R(V)$ is an invertible function. From this we see that, in this case, $End_R(V)$ is a division ring, for if $f \in End_R(V)$ is an invertible function, then $f^{-1} \in End_R(V)$. We also make use of the following result of Ware and Zelmanowitz [14].

Theorem 2.1. *Let R be a commutative ring and $V \in$ mod-R. Then $End_R(V)$ is a division ring if and only if* $(*)$ *[$Ann_R(V)$ is a nonmaximal prime ideal and V is R-isomorphic to $Q(R/Ann_R(V))$, the quotient field of $R/Ann_R(V)$].*

Our characterization of those $M_R(V)$ that are near-fields now follows.

Theorem 2.2. *Let R be a commutative ring and $V \in$ mod-R. The following are equivalent:*
(1) *$M_R(V)$ is a near-field;*
(2) *condition $(*)$ of the above theorem holds;*
(3) *$End_R(V)$ is a field;*
(4) *$M_R(V)$ is a field.*

Proof. (1) $\Rightarrow$ (2): As we noted above, $M_R(V)$ a near-field implies $End_R(V)$ is a division ring so from the Ware–Zelmanowitz theorem we have (2).

(2) $\Rightarrow$ (3): From $(*)$ one obtains that $End_R(V)$ is ring isomorphic to $Q(R/Ann_R(V))$ and so is a field.

(3) $\Rightarrow$ (4): From (3) we have $(*)$ of Theorem 2.1 and so V is R-isomorphic to $Q(R/Ann_R(V))$. Therefore

$$\begin{aligned} M_R(V) &= M_{R/Ann_R(V)}(V) \cong M_{R/Ann_R(V)}(Q(R/Ann_R(V))) \\ &= M_{Q(R/Ann_R(V))}(Q(R/Ann_R(V))). \end{aligned}$$

This last equality follows from the general argument that if D is a domain with field of quotients $Q(D)$, then $M_D(Q(D)) = M_{Q(D)}(Q(D))$. In fact since $D \subseteq Q(D)$, $M_{Q(D)}(Q(D)) \subseteq M_D(Q(D))$. If $a/b \in Q(D)$, $f \in M_D(Q(D))$, $v \in Q(D)$, we have $bf((a/v)v) = f(av) = av$ so $f((a/b)v) = (a/v)\,f(v)$. Thus we obtain $M_R(V) \cong Q(R/Ann_R(V))$, again a field.

Since (4) $\Rightarrow$ (1) is clear, the result is established. □

Now, as usual, let R be a Noetherian commutative ring and suppose V is a uniform R-module. Then using the Matlis Theorem and the fact that V is uniform we get $V \subseteq E(R/P)$ for some prime ideal P of R. Let R_P denote the localization of R at the prime ideal P. From [5], each $r \in R - P$ determines an automorphism of $E(R/P)$ (via left multiplication) and further $E(R/P)$ is an R_P module and as such is isomorphic to $E(R_P/PR_P)$. From [12] or [5], if $\phi \in End_R(E(R/P))$ and n a positive integer, then there exists $r_n/s_n \in R_P$ such that $\phi(e) = (r_n/s_n)e$, for each $e \in (0 : P^n) = \{v \in E(R/P) \mid P^n v = 0\}$.

Let $x \in V$, $f, g \in M_R(V)$ and let $X = Rx$, the cyclic submodule generated by x. Then $f_{|X}$ and $g_{|X}$ are in $End_R(X)$, so by the injectivity of $E(R/P)$ there exist $\bar{f}, \bar{g} \in End_R(E(R/P))$ such that $\bar{f}_{|X} = f_{|X}$ and $\bar{g}_{|X} = g_{|X}$. So there exist r/s, $r'/s' \in R_P$ such that $f(x) = (r/s)x$ and $g(x) = (r'/s')x$. Thus $gf(x) = (rr'/ss')x = fg(x)$. This gives the next result.

Theorem 2.3. *If V is a uniform R-module then $M_R(V)$ is a commutative ring.*

Let $V \in$ mod-R and suppose every element of V is annihilated by some power of a nonzero prime ideal P of R. We say *$x \in V$ has P-ht n* if $P^n x = 0$ but $P^{n-1}x \neq 0$. We often use this concept of *P-ht* in various places in our investigation. We next show that in the case of finitely generated modules, if $M_R(V)$ is a ring then it must be a commutative ring.

Theorem 2.4. *If V is a finitely generated module then $M_R(V)$ is a ring if and only if $M_R(V)$ is a commutative ring.*

Proof. We suppose $M_R(V)$ is a ring and prove it is commutative by induction on the uniform dimension of V. From the previous theorem, if $u \cdot dim(V) = 1$ then $M_R(V)$ is a commutative ring. Now suppose $M_R(V)$ is a ring and $u \cdot dim(V) \leq n$ implies $M_R(V)$ is commutative, and let W be an R-module with $u \cdot dim(W) = n+1$. Then $E(W) = E(R/P_1) \oplus \cdots \oplus E(R/P_n)$ [9, p. 282, Ex. 5] and without loss of generality we assume that P_{n+1} is maximal among $\{P_1, \ldots, P_{n+1}\}$, i.e., $P_{n+1} \subseteq P_i$ implies $P_{n+1} = P_i$. We also assume for some k, $1 \leq k \leq n+1$, $P_k = P_{k+1} = \cdots = P_{n-1}$ and for $i < k$, $P_i \neq P_{n+1}$.

From the Prime Avoidance Theorem [1, p. 56], there exists $r \in P_{n+1} \backslash (\bigcup_{i=1}^{k-1} P_i)$. It follows from Matlis' Theorem and the fact that W is finitely generated, that we can choose a positive integer m such that $r^m(x_1 + \cdots + x_{n+1}) = (r^m(x_1 + \cdots + x_{k-1})$ or is 0 if $k = 1$, for each $w = x_1 + \cdots + x_{n+1} \in W$, $x_i \in E(R/P_i)$.

Now suppose $M_R(W)$ is not commutative. Then there exist $f, g \in M_R(V)$ such that $H := fg - gf \neq 0$. Let $a_1 + \cdots + a_{n+1} \in W$ be such that $H(a_1 + \cdots + a_{n+1}) = b_1 + \cdots + b_{n+1} \neq 0$.

We claim $M_R(r^m W)$ is a ring (which might be the zero ring). If not, there exist $f_1, g_1, h_1 \in M_R(r^m W)$ and $x = r^m a \in r^m W$ such that $f_1(g_1(x) + h_1(x)) \neq f_1 g_1(x) + f_1 h_1(x)$. We define $\bar{f}_1 \in M_R(W)$ by $\bar{f}_1(w) = f_1(r^m w)$ and similarly extend g_1 to $\bar{g}_1$, h_1 to $\bar{h}_1$. Since multiplication by r^m acts as an isomorphism on $E(R/P_1) \oplus \cdots \oplus E(R/P_{k-1})$ [5] and $r^m W \subseteq E(R/P_1) \oplus \cdots \oplus E(R/P_{k-1})$, we have $r^m f_1(g_1(x) + h_1(x)) \neq r^m(f_1 g_1(x) + f_1 h_1(x))$. On the other hand, $\bar{f}_1(\bar{g}_1(a) + \bar{h}_1(a)) = \bar{f}_1\bar{g}_1(a) + \bar{f}_1\bar{h}_1(a)$ since $M_R(V)$ is a ring and this in turn implies $r^m f_1(g_1(x) + h_1(x)) = r^m(f_1 g_1(x) + f_1 h_1(x))$. This contradiction shows that we must have $M_R(r^m W)$ a ring.

Now $f_{|r^m W}, g_{|r^m W} \in M_R(r^m W)$, $u \cdot dim(r^m W) \leq n$ and $M_R(r^m W)$ a ring implies, by the induction hypothesis, that $H_{|r^m W} = 0$. So $0 = H(r^m a_1 + \cdots + r^m a_{k-1}) = H(r^m(a_1 + \cdots + a_{n+1}) = r^m(b_1 + \cdots + b_{k-1}) = r^m b_1 + \cdots + r^m b_{k-1}$ so $b_1 = b_2 = \cdots = b_{k-1} = 0$, again since r^m acts as an isomorphism on $E(R/P_1) \oplus \cdots \oplus E(R/P_{k-1})$.

Thus if there exists another prime ideal among the P_i that is maximal (but not equal to P_{n+1}), then using the above arguments we would find $H = 0$ and $M_R(W)$ commutative. So we now assume that $P_i \subseteq P_{n+1} =: Q$ for $i = 1, \ldots, n$. Hence we can regard $E(W)$ as a R_Q-module. If $Q = \{0\}$, V is torsion-free and the result follows from [13, Theorem 4.1]. Thus we take $Q \neq \{0\}$.

Let $W^* = W \cap (E(R/P_k) \oplus \cdots \oplus E(R/P_{n+1})$ and from above we have $H(W) \subseteq W^*$ and each element in W^* is annihilated by some power of Q. We next choose $\alpha \in W$ such that $H(w)$ has maximal Q-ht (recall W is finitely generated). Note further that $H(\alpha) \neq 0$ implies $Q\text{-}ht(H(\alpha)) \geq 1$. Let $X = R_Q\alpha \cap W$ and $s \in Q^{Q\text{-}ht(H(\alpha))-1}$ such that $Q\text{-}ht(sH(\alpha)) = 1$. Define $\ell : W \to W$ by

$$\ell(w) = \begin{cases} sH(w) & \text{if } w \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We show $\ell \in M_R(W)$. First, if $w \in X$ then $tw \in X$ so $t\ell(w) = tsH(w) = sH(tw) = \ell(tw)$. Next, if $w \notin X$ and $t \in Q$, then, from the maximality of $H(\alpha)$, we see that $Q\text{-}ht(sH(w)) \leq 1$ so for $t \in Q$, $tsH(w) = 0$. From this we see $\ell(tw) = t\ell(w)$. Finally, if $w \notin X$ and $t \notin Q$ then $rw \notin X$ since $rw \in X \Rightarrow w = (1/r)(rw) \in X$. Therefore $\ell \in M_R(W)$.

If $f(X) \subseteq X$ and $g(X) \subseteq X$, say $f(\alpha) = (r_1/s_1)\alpha$ and $g(\alpha) = (r_2/s_2)\alpha$ then $gf(\alpha) = fg(\alpha)$ so $H(\alpha) = 0$, contrary to the choice of α. So without loss of generality we take $f(X) \nsubseteq X$ which means $f(\alpha) \notin X$. Consequently, $\ell(1 + f)(\alpha) = \ell(\alpha + f(\alpha)) = 0$ since $\alpha + f(\alpha) \notin X$ while $(\ell \cdot 1 + \ell \cdot f)\alpha = \ell(\alpha) \neq 0$, contrary to the fact that $M_R(W)$ is a ring. Thus we conclude that $H = 0$, i.e. $fg = gf$ for all $f, g \in M_R(V)$. □

We next obtain another sufficient condition for $M_R(V)$ to be a ring, in fact a commutative ring. We will say V *has no containment* when the injective hull of $V, E(V) = \bigoplus_{i \in I} E(R/P_i)$ is such that $P_i \nsubseteq P_j$ if $i \neq j$.

Theorem 2.5. *Let $V \subseteq \bigoplus_{i \in I} E(R/P_i)$ such that V has no containment. Then $M_R(V)$ is a commutative ring.*

Proof. We show $M_R(V)$ is commutative, hence a commutative ring. Suppose the contrary. Then there exist $f, g \in M_R(V)$ and $x \in V$ such that $(gf - fg)(x) \neq 0$. Let $x_{j_1}, \ldots, x_{j_n}$ be the nonzero components of x and $y_{k_1}, \ldots, y_{k_m}$ the nonzero components of $(gf - fg)(x)$. Using Lemma 3.55 of [11], choose

$$s \in \left(\left(\bigcap_{i \geq 2} P_{k_i} \right) \cap \left(\bigcap_{P_{j_i} \neq P_{k_1}} P_{j_i} \right) \right) \setminus P_{k_1}$$

(where we take $s = 1$ if $m = n = 1$ and $j_2 = k_1$), and let $N = \max\{P_{j_1}\text{-}htx_{j_1}, \ldots, P_{j_n}\text{-}htx_{j_n}, P_{k_1}\text{-}htx_{k_1}, \ldots, P_{k_m}\text{-}htx_{k_m}\}$. Then since s acts as an isomorphism on $E(R/P_{k_1})$, the only nonzero component of $s^N(gf - fg)(x)$ is $s^N y_{k_1}$. But then $0 \neq s^N y_{k_1} = s^N(gf - fg)(x) = (gf - fg)(s^N x)$ which means $s^N x \neq 0$ so some component of x must be in P_{k_1}, say $j_1 = k_1$, i.e., $0 \neq s^N y_{k_1} = (gf - fg)s^N x_{j_1}$.

Let $W = V \cap E(R/P_{k_1})$. If we can show $h(W) \in W$ for all $h \in M_R(V)$, then since $s^N x_{j_1} \in W$, we have $gf \neq fg$ in $M_R(W)$. But W is uniform, so from Theorem 2.3, $M_R(W)$ must be a commutative ring. This contradiction will give us the desired result. To this end suppose $w \in W$ and $h(w) \notin W$ for some $h \in M_R(V)$. Then $h(w) \neq 0$ and $h(w)$ must have nonzero components other than in $E(R/P_{k_1})$. We let $w_{\ell_1}, \ldots, w_{\ell_d}$ be the nonzero components of $h(w)$ and without loss of generality assume $\ell_1 \neq k_1$. Let $t \in P_{k_1} \setminus P_{\ell_1}$ and let $M \geq P_{k_1}$-$ht\, w$. Then $0 = h(0) = h(t^M w) = t^M h(w) = t^M w_{\ell_1} + \cdots + t^M w_{\ell_d} \neq 0$ since $t \notin P_{\ell_1}$ and multiplication by t acts as an isomorphism in $E(R/P_{\ell_1})$. This contradiction shows that we must have $h(W) \subseteq W$ for all $h \in M_R(V)$. □

We show that the converse of the above theorem is true for injective modules. However, we first give a characterization of "no containment" in terms of properties of R and V.

Theorem 2.6. *Let $V \in mod$-R. Then $V \subseteq \bigoplus_{i \in I} E(R/P_i)$ with $P_i \subseteq P_j$ for $i \neq j$ if and only if for $x, y \in V^* = V - \{0\}$, $Rx \cap Ry = \{0\}$ implies $Ann_R(x) \not\subseteq Ann_R(y)$.*

Proof. First note that $V \subseteq \bigoplus_{i \in I} E(R/P_i)$ if and only if $E(V) = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ with $P_\lambda \subseteq P_{\lambda'}$ for $\lambda \neq \lambda'$ [9], where $\Lambda \subseteq I$.

Suppose first that $E(V) = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ with $P_\lambda \not\subseteq P_{\lambda'}$ for $\lambda \neq \lambda'$. Let $x, y \in V^*$ with $Rx \cap Ry = \{0\}$, say $x = x_1 + \cdots + x_n$ with $x_i \in E(R/P_{\lambda_i})^*$ and $y = y_r + \cdots + y_m$ with $y_j \in E(R/P_{\lambda_j})^*$. From [11, Lemma 3.55], we find there exist

$$s_j \in \left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} P_i \right) \Big\backslash P_j,$$

hence there exist n_i such that $s_j^{n_i} x_i = 0$ for $i \neq j$ but $s_j^{n_j} x_j \neq 0$. From this $Rx \cap E(R/P_{\lambda_i}) \neq \{0\}$ for $i = 1, 2, \ldots, n$ and, similarly, $Ry \cap E(R/P_{\lambda_j}) \neq \{0\}$ for $j = r, \ldots, m$. Since $Rx \cap Ry = \{0\}$ we have $r > n$.

We now claim $\sqrt{Ann_R(X_i)} = R_{\lambda_i}$. In fact we show that, if $W \subseteq E(R/P)$, then for any $w \in W^*$, $\sqrt{Ann_R(w)} = P$. We know $P^m w = 0$ for some m, so $P \subseteq \sqrt{Ann_k w}$. Conversely, if $r \in \sqrt{Ann_R(w)}$ then $r^m \in Ann_R(w)$ for some m. But then $r^m \in P$ since elements not in P act as isomorphisms on $E(R/P)$. But then $r \in P$.

Assume $Ann_R x \subseteq Ann_R y$ and note $Ann_R(x) = \bigcap_{i=1}^n Ann_R(x_i)$, $Ann_R(y) = \bigcap_{j=r}^m Ann_R(y_j)$. Then

$$\bigcap_{i=1}^{n} P_{\lambda_i} = \bigcap_{i=1}^{n} \sqrt{Ann_R(x_i)} = \sqrt{\bigcap_{i=1}^{n} Ann_R(x_i)} \subseteq \sqrt{\bigcap_{j=r}^{m} Ann_R(y_j)}$$
$$= \bigcap_{j=1}^{m} \sqrt{Ann_R(y_j)} = \bigcap_{j=r}^{m} P_{\lambda_j} \subseteq P_{\lambda_m}.$$

But $P_{\lambda_m} \supseteq \bigcap_{i=1}^n P_{\lambda_i}$ implies $P_{\lambda_m} \supseteq P_{\lambda_i}$ for some i, [11, Lemma 3.55], a contradiction, so we must have $Ann_R(x) \not\subseteq Ann_R(y)$.

For the converse, suppose $E(V) = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ but $P_{\lambda_1} \leq P_{\lambda_2}$ for some $\lambda_1 \neq \lambda_2$. Let x be nonzero in $V \cap E(R/P_{\lambda_1})$ and y nonzero in $V \cap E(R/P_{\lambda_2})$. Since $P_{\lambda_1}^m x = 0$ for some m, there exists $r \in R$ such that $0 \neq rx \in Ann_R(P_{\lambda_1})$ so $P_{\lambda_1} \subseteq Ann_R(rx)$. Again, since elements not in P_{λ_1} act as isomorphisms on $E(R/P_{\lambda_1})$ and $rx \in E(R/P_{\lambda_1})$ we must have $Ann_R(rx) = P_{\lambda_1}$. Similarly, there exists $s \in R$ such that $Ann_R(sy) = P_{\lambda_2}$.

Since $x \in E(R/P_{\lambda_1})$ and $y \in E(R/P_{\lambda_2})$, $Rrx \cap Rsy = \{0\}$, but $Ann_R(rx) = P_{\lambda_1} \subseteq P_{\lambda_2} = Ann_R(sy)$, contrary to our hypothesis. Hence the result. □

Theorem 2.7. *Let V be an injective R-module, $V = \bigoplus_{i \in I} E(R/P_i)$. The following are equivalent:*

(1) *$M_R(V)$ is a commutative ring;*

(2) *V has no containment;*

(3) *If $Rx \cap Ry = \{0\}$ for $x, y \in V^*$, then $Ann_R(x) \nsubseteq Ann_R(y)$;*

(4) *$End_R(V)$ is a commutative ring.*

Proof. From the above theorem, (2) ⇔ (3), from Theorem 2.5, (2) ⇒ (1) and clearly (1) ⇒ (4). It remains to show (4) ⇒ (2). To this end suppose there is some containment, say $P_1 \subseteq P_2$. We show $End_R W$ is non-commutative where $W = E(R/P_1) \oplus E(R/P_2)$. But this in turn implies $End_R(V)$ is not commutative contradicting (4). Now

$$End_R W \cong \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \middle| \begin{array}{l} \alpha \in End_R(E(R/P_1)), \beta \in Hom_R(E(R/P_1), E(R/P_2)), \\ \gamma \in Hom_R(E(R/P_2), E(R/P_1)), \delta \in End_R(E(R/P_2)) \end{array} \right\}.$$

From [12, Prop. 4.21], $P_1 \subseteq P_2$ implies there exists a nonzero $\beta \in Hom_R(E(R/P_1), E(R/P_2))$. But then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

hence the result. □

Corollary 2.8. *Let V be a finitely generated injective module. Then $M_R(V)$ is a ring if and only if V satisfies condition* (3) *of the above theorem.*

Proof. Theorems 2.7 and 2.4. □

Corollary 2.9. *If R is Artinian and V injective then $M_R(V)$ is a ring if and only if V satisfies condition* (3) *of the above theorem.*

Proof. Since R is Artinian, every prime ideal is maximal and there are only a finite number of maximal ideals [1, p. 89] and so $V = \bigoplus_{i \in I} E(R/M_i)$. Also each $E(R/M_i)$ is finitely generated [5, Theorem 3.11]. If I is infinite then for some $i \neq j$, $M_i = M_j$. Let $W = E(R/M_i) \oplus E(R/M_j)$ and note that W is finitely generated. Since $V = W \oplus W'$ for some submodule W' of V, every function $f \in M_R(W)$ can be extended to a function $\bar{f} \in M_R(V)$ ($\bar{f}(w+w') = f(w)$, $w \in W$, $w' \in W'$) and so $M_R(W)$ can be embedded in $M_R(V)$. Consequently, if $M_R(V)$ is a ring so is $M_R(W)$ and, by Corollary 2.8, condition (3) holds for W. Since $M_i = M_j$ this is a contradiction which means that I must be finite, hence V is finitely generated. Thus when $M_R(V)$ is a ring, condition (3) holds for V. The converse follows from Theorem 2.7. □

We next give an example to show that, without some conditions on V, $End_R V$ can be a commutative ring and $M_R(V)$ need not even be a ring.

Example 2.10. Let $R = \mathbb{Z}_2[x,y]/(x^3, x^2y, xy^2, x^3)$ and $V = (x,y)/(x^3, x^2y, xy^2, y^3)$. We use representations of classes to denote the elements of both R and V. We observe first that, if $\varphi \in End_R V$, then the term y does not occur in $\varphi(x)$. For if so, then y^2 occurs in $\varphi(xy)$ which contradicts $x\varphi(y) = \varphi(xy)$. Similarly no x appears in $\varphi(y)$.

Suppose now $f, g \in End_R(V)$ with $f(x) = \alpha_1 x + \beta_1 y^2$, $g(x) = \alpha_2 x + \beta_2 y^2$, $f(y) = \gamma_1 y + \delta_1 x^2$, $g(y) = \gamma_2 y + \delta_2 x^2$ where $\alpha_i, \beta_i, \gamma_i, \delta_i \in R$. Let $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$, $\bar{\delta}_i$ denote the constant terms of the $\alpha_i, \beta_i, \gamma_i, \delta_i$ respectively. Then $xf(y) = yf(x)$ implies $\bar{\alpha}_1 = \bar{\gamma}_1$ and in a similar manner we get $\bar{\alpha}_2 = \bar{\gamma}_2$. To show $End_R(V)$ is commutative it suffices to show $gf(x) = fg(x)$ and $gf(y) = fg(y)$. Now,

$$\begin{aligned} gf(x) &= g(\alpha_1 x + \beta_1 y^2) = \alpha_1\alpha_2 x + \alpha_1\beta_2 y^2 + \beta_1(\gamma_2 y + \delta_2 x^2)^2 \\ &= \alpha_1\alpha_2 x + (\alpha_1\beta_2 + \beta_1\gamma_2^2)y^2 \\ &= \alpha_1\alpha_2 x + (\bar{\alpha}_1\bar{\beta}_2 + \bar{\beta}_1\bar{\gamma}_2)y^2 = \alpha_1\alpha_2 x + (\bar{\alpha}_1\bar{\beta}_2 + \bar{\beta}_1\bar{\alpha}_2)y^2. \end{aligned}$$

Also,

$$\begin{aligned} fg(x) &= f(\alpha_2 x + \beta_2 y^2) = \alpha_1\alpha_2 x + \alpha_2\beta_1 y^2 + \beta_2(\gamma_1 y + \delta_1 x^2)^2 \\ &= \alpha_1\alpha_2 x + (\alpha_2\beta_1 + \beta_2\gamma_1^2)y^2 = \alpha_1\alpha_2 x + (\bar{\alpha}_2\bar{\beta}_1 + \bar{\beta}_2\bar{\gamma}_1)y^2 \\ &= \alpha_1\alpha_2 x + (\bar{\alpha}_2\bar{\beta}_1 + \bar{\beta}_2\bar{\alpha}_1)y^2 = gf(x). \end{aligned}$$

In the same manner, $gf(y) = fg(y)$ so $End_R V$ is a commutative ring. Define $\varphi \in End_R(V)$ by $\varphi(x) = y^2$ and $\varphi(y) = 0$ and so $\varphi(x) \notin Rx$. Since R is a finite local ring, from [7, Theorem 4.2], $M_R(V)$ is not a ring.

If R is an Artinian ring, then from [5, Theorem 3.11], every indecomposable injective module is finitely generated. Thus if $V \subseteq \bigoplus_{i=1}^{n} E(R/P_i)$ then V is finitely generated. An application of Theorem 2.4 yields the next result.

Theorem 2.11. *Let R be an Artinian ring and let V be an R-module with finite uniform dimension. Then $M_R(V)$ is a ring if and only if $M_R(V)$ is a commutative ring.*

From [13, Theorem 4.2], $M_R(V)$ is a ring if and only if $M_R(V)$ is a commutative ring when R is a Dedekind domain. An application of Theorem 2.7 gives our next result.

Theorem 2.12. *Let D be a Dedekind domain and V an injective D-module. The following are equivalent*:

(1) *$M_R(V)$ is a ring*;

(2) *$M_R(V)$ is a commutative ring*;

(3) *V has no containment*;

(4) *$End_R(V)$ is a commutative ring.*

We have seen above that in several instances $M_R(V)$ is a ring if and only if it is commutative and this is equivalent to no containment. It may be that there is containment and $M_R(V)$ is still a ring. In Theorem 2.17 we show that when all summands of V are equal, $M_R(V)$ is a ring precisely when it is $End_R(V)$. In Section 3 we give specific examples of injective modules V with containment such that $M_R(V)$ is a ring with $M_R(V) = End_R(V)$. On the other hand, there are many situations in which V having containment implies that $M_R(V)$ is not a ring. We now investigate this situation further.

Recall that if W is an R-module with a non-trivial direct decomposition and $M_R(W)$ is a ring, then W is R-connected. (See [8] for the definition and remarks about connectedness.)

Lemma 2.13. *Let* $V \subseteq E(R/P) \oplus E(R/P)$, *where* P *is a nonzero principal ideal, say* $P = (p)$. *If* $X = \{a + b \in V \mid P\text{-}ht(a) = P\text{-}ht(b)\}$ *then* X *is a union of components of* V.

Proof. Note that $0 \in X$ so $X \neq \emptyset$. Note also that for $a \in E(R/P)$, the $P\text{-}ht(a)$ is the least nonnegative integer n such that $p^n a = 0$. For suppose $P\text{-}ht(a) = m$ so m is the least nonnegative integer such that $P^m a = 0$. But $p \in P$ implies $p^m a = 0$. But, $p^{\overline{m}} a = 0$ implies $P^{\overline{m}} a = 0$ so $\overline{m} \geq m$.

Now take $a+b \in X$, $r \in R$ with (say), $P\text{-}ht(a) = P\text{-}ht(b) = \alpha$. We show $r(a+b) \in X$.

Case (i): $r \notin P$. Then since multiplication by r induces an isomorphism on $E(R/P)$, $P\text{-}ht(ra) = P\text{-}ht(a) = P\text{-}ht(b) = P\text{-}ht(rb)$.

Case (ii): $r \in \bigcap_{n=1}^{\infty} P^n$. Then $ra = rb = 0$.

Case (iii): $r \in P^m \backslash P^{m+1}$. Then $r = sp^m$, $s \notin P$. If $\alpha \leq m$, $ra = rb = 0$. If $\alpha > m$, $P\text{-}ht(ra) = P\text{-}ht(p^m a) = \alpha - m$ since $p^{\alpha-m} p^m a = p^\alpha a = 0$ and if $\ell < \alpha - m$, $p^\ell p^m a = p^{\ell+m} a \neq 0$ since $\ell + m < \alpha$. Similarly, $P\text{-}ht(rb) = \alpha - m$.

We now take $r(a+b) \in X^*$, $r \in R$, $a + b \in V$ and show $a + b \in X$. If $r \notin P$ then as above $a + b \in X^*$. The case $r \in \bigcap_{n=1}^{\infty} P^n$ cannot occur since $r(a + b) \neq 0$. Now suppose $r \in P^m \backslash P^{m+1}$. If $m > P\text{-}ht(a)$ and $m > P\text{-}ht(b)$ then $r(a + b) = 0$ which is impossible. If $m > P\text{-}ht(a)$ and $m < P\text{-}ht(b)$ then $P\text{-}ht(ra) \neq P\text{-}ht(rb)$, again impossible since $ra + rb \in X^*$. Thus we must have $m < P\text{-}ht(a)$ and $m < P\text{-}ht(b)$. But, as above, we then get $P\text{-}ht(ra) = P\text{-}ht(a) - m$ and $P\text{-}ht(rb) = P\text{-}ht(b) - m$. Hence $P\text{-}ht(a) = P\text{-}ht(b)$.

From these observations we see that if $x \in X$, then the R-connected component determined by x is contained in X. □

Corollary 2.14. *If* $V \subseteq E(V) = E(R/P) \oplus E(R/P)$ *where* $P = (p)$ *is principal then* V *is not R-connected.*

Proof. Let $0 \neq a \in V \cap (E(R/P) \oplus \{0\})$ and $0 \neq b \in V \cap (\{0\} \oplus E(R/P))$. If $P = \{0\}$ then $a \in Q(R) \oplus \{0\}$, $b \in \{0\} \oplus Q(R)$ and since R is then a domain we see V is not connected.

Thus we take $P \neq \{0\}$. Then using Matlis' Theorem, there exist $r, s \in R$ such that $P\text{-}ht(ra) = P\text{-}ht(sb) = 1$. So X, as defined in the above lemma, is nonzero since $ra+sb \in X$. Since $a = a+0 \notin X$, we have $\{0\} \subsetneq X \subsetneq V$, i.e., V is not R-connected. □

We note that the above lemma and its corollary hold for arbitrary direct sums.

Theorem 2.15. *Let* $V = \bigoplus_{i \in I} E(R/P_i)$ *with* $P_i = (P) = P_j$ *for some* $i \neq j$. *Then* $M_R(V)$ *is not a ring.*

Proof. Without loss of generality we suppose $P_1 = (p) = P_2$ and let $W = E(R/P_1) \oplus E(R/P_2)$. As we saw in the proof of Corollary 2.9, $M_R(W)$ can be embedded in $M_R(V)$. Thus it suffices to show $M_R(W)$ is not a ring. But this follows from the above corollary and the remarks preceding the lemma. □

Let S be a local ring with unique maximal ideal J and suppose J is principal, say $J = (a)$. From Krull's Principal Ideal Theorem [11, Theorem 15.2], $ht(J) \leq 1$. Now let P be a prime ideal of S, $P \subsetneq J$. For $b \in P$, $b = b_1 a$, for some $b_1 \in S$. Since $a \notin P$, $b_1 \in P$ so $b_1 = b_2 a$, hence $b = b_2 a^2$. By induction $b \in J^n$, $n \geq 1$, consequently $b \in \bigcap_{n=1}^{\infty} J^n$. But by the Krull Intersection Theorem [11, Corollary 8.25], $\bigcap_{n=1}^{\infty} J^n = \{0\}$, so $P = \{0\}$. We use these observations in our next result.

Theorem 2.16. *Let* $V = \bigoplus_{i \in I} E(R/P_i)$ *with containment among the primes* P_i. *Let* R *have the property that localization at a maximal ideal* M *results in a principal ideal* MR_M. *Then* $M_R(V)$ *is not a ring.*

Proof. Again, without loss of generality we take $P_1 \subseteq P_2$ and consider $W = E(R/P_1) \oplus E(R/P_2)$. As above, it suffices to show $M_R(W)$ is not a ring.

Let M be a maximal ideal in R containing P_2. By hypothesis MR_M is principal. Then the bijection between $\{P \in \operatorname{Spec}(R) \mid P \subseteq M\}$ and $\{P \in \operatorname{Spec}(R_M)\}$ $(P \mapsto PR_M)$ gives $P_1 R_M \subseteq P_2 R_M \subseteq MR_M$. From the remarks above and the bijection, we obtain the following possibilities, $P_1 = P_2 = \{0\}$, $\{0\} = P_1 \subsetneq P_2 = M$ or $P_1 = P_2 = M$.

If $P_1 = P_2 = \{0\}$ then $W = E(R/\{0\}) \oplus E(R/\{0\}) = Q(R) \oplus Q(R)$, where $Q(R)$ is the field of quotients of R. But then $M_R(W) = M_{Q(R)}(W)$ which is *not* a ring.

If $\{0\} = P_1 \subsetneq P_2 = M$, $W = Q(R) \oplus E(R/M)$, which has both torsion and torsion-free elements. Thus W is not connected and again $M_R(W)$ is not a ring.

Finally, if $P_1 = P_2 = M$, $W = E(R/M) \oplus E(R/M)$ and is R_M isomorphic to $\widehat{W} = E(R_M/MR_M) \oplus E(R_M/MR_M)$ [12, Prop. 5.6], i.e., $M_R(W) \cong M_{R_M}(\widehat{W})$. From Corollary 2.14, $M_{R_M}(\widehat{M})$ is not a ring, therefore the result is established.

Since Dedekind domains R have the property of the above theorem, we obtain an alternate proof of (1) ⇒ (3) in Theorem 2.12.

As a final situation with containment we suppose V is a finite sum of $E(R/P_i)$ in which all the P_i are equal. Here we find $M_R(V)$ is a ring if and only if it is the ring of endomorphisms of V. □

Theorem 2.17. *Let $V = \bigoplus_{i \in I} W_i$ where $W_i = W$, for all $i \in I$ and $|I| \geq 2$. Then $M_R(V)$ is a ring if and only if $M_R(V) = End_R(V)$.*

Proof. Let $\pi_i : V \to W_i$ and $\varepsilon_i : W_i \to V$ be the natural projection and injection maps respectively. Suppose $M_R(V)$ is a ring. Then for all $f \in M_R(V)$, $f = \sum_i f\varepsilon_i\pi_i$ so, for $v \in V$, $v = \sum_i v_i$, $v_i = \varepsilon_i\pi_i v$, hence $f(v) = \sum_i f(v_i)$. Now let $a = \sum_i a_i$, $b = \sum_i b_i$ be elements of V and let $g \in M_R(V)$. Define $g_{ij} = g(\varepsilon_i\pi_i + \varepsilon_{ij}\pi_j) \in M_R(V)$ where ε_{ij} takes $x \in W_j$ and injects it into the ith position in V. Therefore, for $i \neq j$, $g(a_i + b_i) = g_{ij}(a_i + \varepsilon_{ji}\pi_i b_i) = g_{ij}(a_i) + g_{ij}\varepsilon_{ji}\pi_i b_i$ (since $\varepsilon_{ji}\pi_i b_i$ is in the jth position) $= g(a_i) + g(b_i)$. Thus $g(a+b) = \sum_i g(a_i+b) = \sum_i (g(a_i)+g(b_i)) = \sum_i g(a_i) + \sum_i g(b_i) = g(a) + g(b)$. Hence $M_R(V) \subseteq End_R(V)$ and the rest is clear. □

3. A class of examples

In this section we consider certain modules over the ring $K[x_1, \ldots, x_n]$ where K is a field. We use these modules to determine some injective hulls. We thus provide methods for constructing examples of injective hulls other than the standard constructions modeled after Dedekind domains. We start with two general results to be used in our development.

Theorem 3.1. *Let $V \in$ mod-R. Then $V = E(R/P)$ if and only if*

(a) *V is uniform,*
(b) *$\sqrt{Ann_R(v)} = P$ for some $v \in V^*$,*
(c) *$V = V_P$,*
(d) $\dim_{R_P/PR_P}(B_n/B_{n-1}) = \dim_{R_P/PR_P} \frac{(PR_P)^{n-1}}{(PR_P)^n}$ *where $B_n = (0 :_V P^n)$, $n = 1, 2, 3, \ldots$*

Proof. "⇒" Since R/P is a uniform R-module and $E(R/P)$ is an essential extension of R/P, $E(R/P)$ is uniform, hence we have (a), and (b) was established in the proof of the Theorem 2.6. For (c) note that $s \notin P$ and $v \in V^*$ implies $sv \neq 0$. Therefore $\phi : V \to V$, $\phi(v) = v/1$ is injective and consequently V can be identified with $V' = \{v/1 \mid v \in V\} \subseteq V_P$. Since $v/s = ss^{-1}v/s = s^{-1}v/1$, we have $V = V_P$. For (d), we first note that $C_n/C_{n-1} \cong \frac{(PR_P)^{n-1}}{(PR_P)^n}$ as R_P/PR_P-vector spaces where $C_n = (0 :_{E(R_P/PR_P)} (PR_P)^n)$. (See [12], paragraph prior to Lemma 5.11.) But since $B_n = (0 :_{E(R/P)} P^n) = (0 :_{E(R/P)} (PR_P)^n)$ and $E(R/P) \cong E(RP/PR_P)$ as R_P-modules [12, Prop 5.6], we have $\dim_{R_P/PR_P}(B^n/B^{n-1}) = \dim_{R_P/PR_P} \frac{(PR_P)^{n-1}}{(PR_P)^n}$.

"⇐" From (c) we see that V can be regarded as an R_P-module. Further, $\sqrt{Ann_R(v)} = P$ implies $\sqrt{Ann_{R_P}(v)} = PR_P$ since

$$\left(\frac{r}{s}\right)^n v = 0 \Leftrightarrow \frac{r^n}{s^n} \cdot \frac{v}{1} = 0 \Leftrightarrow \frac{r^n v}{s^n} = 0 \Leftrightarrow r^n v = 0.$$

Moreover, V uniform in mod-R implies V is uniform in mod-R_P, hence ${}_{R_P}V \subseteq E(R_P/PR_P)$ and $B_n = (0 :_{{}_R V} P^n) = (0 :_{R_P V} PR^P)$. For ease of notation we let $A = R_P$ and $J =$

PR_P. Therefore we have $V \subseteq E(A/J)$, $B_n = (0:_V J^n)$ and from (d), $dim_{A/J} B_n/B_{n-1} = dim_{A/J} \frac{J^{n-1}}{J^n}$. Let $A_n = (0:_{E(A/J)} J^n)$. By induction we show $B_n = A_n$ for all n and since $E(A/J) = \bigcup_{n=1}^{\infty} A_n$ we will have $E(R/P) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = E(A/J)$. Since $V \subseteq E(A/J)$ and $\bigcup_{n=1}^{\infty} A_n \subseteq V$, the result follows.

Clearly $B_0 = A_0$ so we assume $B_k = A_k$ but $B_{k+1} \subsetneq A_{k+1}$. Let $x \in A_{k+1} - B_{k+1}$. Then $x + A_k \notin B_{k+1}/A_k \subseteq A_{k+1}/A_k$. This in turn implies that $dim_{A/J} B_{k+1}/B_k = dim_{A/J} B_{k-1}/A_k < dim_{A/J} A_{k-1}/A_k = dim_{A/J} J^{n-1}/J^n$, a contradiction. Hence the result follows. □

Theorem 3.2. *Let $V \in$ mod-R and let M be a maximal ideal of R. Then $V = E(R/M)$ if and only if*

(a) *V is uniform,*

(b) *$\sqrt{Ann_R(v)} = M$ for some $v \in M^*$,*

(c) *$dim_{R/M}(B_n/B_{n-1}) = dim_{R/M}(\frac{M^{n-1}}{M^n})$ where B_n is defined above.*

Proof. From the previous theorem, it suffices to verify

(i) $dim_{R/M} B_n/B_{n-1} = dim_{R_M/MR_M}(\frac{B^n}{B^{n-1}})$,

(ii) $dim_{R/M} \frac{M^{n-1}}{M^n} = dim_{R_M/MR_M}(MR_M)^{n-1}/(MR_M)^n$ and

(iii) $V = V_M$ if $V \subseteq E(R/M)$.

We note that (i) follows from the fact that the map $r + M \mapsto r/1 + MR_M$ is an isomorphism. In fact, if $r/1 + MR_M = 0$, then $r/1 = m/s$ which in turn implies $tsr = tm$ for some $t \notin M$ and so $r \in M$. To verify that the map is surjective, it suffices to show that $1/s + MR_M$ is an image for $s \notin M$. But $s \notin M$ implies there is some $r \in R$, $m \in M$ such that $rs + m = 1$. So

$$\frac{1}{s} + MR_M = \left(\frac{rs+m}{1} + MR_M\right)\left(\frac{1}{s} + MR_M\right) = \frac{r}{1} + MR_M.$$

For (ii) suppose $\{m_1 + M^n, \dots, m_k + M^n\}$ is linearly independent over R/M. Then $\{m_1/1 + (MR_M)^n, \dots, m_k/1 + (MR_M)^n\}$ is linearly independent over $\frac{R_M}{MR_M}$. If not, $\sum_{i=1}^k (r_i/s_i + MR_M)(m_i/1 + (MR_M)^n) = 0$ so $\sum_{i=1}^k r_i m_i/s_i = m/t$, $m \in M^n$, $t \notin M$. Hence $t'(t\sum_{i=1}^k q_i r_i m_i - qm) = 0$, $t' \notin M$, $q_i = \prod_{j \neq i} s_j$, $q = \prod_{j=1}^k$ and so $\sum_{i=1}^k t' t q_i r_i m_i \in M^n$. But this implies $m_1 + M^n, \dots, m_k + M^n$ are linearly dependent over R/M.

Conversely, if $\{m_1/s_1 + (MR_M)^n, \dots, m_k/s_k + (MR_M)^n\}$ is linearly independent then so is $\{m_1 + M^n, \dots, m_k + M^n\}$. If not, then we have $\sum_{i=1}^k (r_i + M)(m_i + M^n) = 0$ or $\sum_{i=1}^k r_i m_i \in M^n$. From this we get

$$\sum_{i=1}^k \frac{r_i}{1}\frac{m_i}{1} \in (MR_M)^n \text{ so } \sum_{i=1}^k \left(\frac{r_i s_i}{1} + MR_M\right)\left(\frac{m_i}{s_i} + (MR_M)^n\right) = 0,$$

a contradiction.

Finally for (iii) let $v/s \in V_M$. Then $s \notin M$ implies there exist $r \in R$, $m \in M$ such that $rs + m = 1$. We choose ℓ such that $m^\ell v = 0$ and find $v/s = ((rs+m)/s)^\ell v = w/1$ for some $w \in V$. Therefore $V = V_M$. □

We apply the above theorem to show that certain modules are uniform injective. Let $R = K[x, y]$ where K is a field and let

$$V = \frac{\langle x^i y^j \mid i, j \in \mathbb{Z}\rangle}{\langle x^i y^j \mid i \geq 0 \text{ or } j \geq 0\rangle},$$

where $\langle x^i y^j \mid i, j \in \mathbb{Z}\rangle$ is the R-submodule of $K(x, y)$ generated by $\{x^i y^j \mid i, j \in \mathbb{Z}\}$ and $\langle x^i y^j \mid i \geq 0 \text{ or } j \geq 0\rangle$ is the R-submodule of $K(x, y)$ generated by $\{x^i y^j \mid i \geq 0 \text{ or } j \geq 0\}$. Each nonzero element of V can be written uniquely in the form $\sum_{\ell=1}^{m} \alpha_\ell x^{\ell_1} y^{\ell_2}$, $\alpha_\ell \in K$, $\ell_i < 0$, $i = 1, 2$. It is clear that V is uniform since each nonzero submodule of V contains $x^{-1}y^{-1}$. Moreover, $\sqrt{(Ann_R x^{-1} y^{-1})} = \langle x, y\rangle = M$(say), a maximal ideal of R, and so $V \subseteq E(R/M)$. We next show that condition (c) of the above theorem holds, consequently we will have $V = E(R/M)$.

We note first that M^{d-1} is generated by elements of the form $x^a y^b$ where $a+b = d-1$ and so the images of these elements will give a basis for the $K(= R/M)$-vector space M^{d-1}/M^d. Let $B_d = (0 :_V M^d)$ and $B_d^- = \{x^a y^b \mid a + b \geq -d\}$. We show $B_d = B_{d+1}^-$. Let $x^a y^b \in M^d$ and $x^{\bar{a}} y^{\bar{b}} \in B_{d+1}^-$. We have $(x^a y^b)(x^{\bar{a}} y^{\bar{b}}) = 0$. Otherwise, $a + \bar{a} \leq -1$ and $b + \bar{b} \leq -1$ which means $\bar{a} + \bar{b} \leq -1 - a - 1 - b = -(a + b) - 2 \leq -d - 2 = -(d + 2)$. But this contradicts $x^{\bar{a}} y^{\bar{b}} \in B_{d+1}^-$, hence we must have $B_{d+1}^- \subseteq B_d$. Suppose $x^a y^b \in B_d \backslash B_{d+1}^-$, so $a + b < -(d + 1)$. From this we see that $x^{-a-1} y^{-b-1} \in M^d$ since $a \leq -1$, $b \leq -1$ and $-a - 1 - b - 1 = -(a + b) - 2 > (d + 1) - 2 = d - 1$. But then $(x^{-a-1} y^{-b-1})(x^a y^b) = x^{-1} y^{-1} \neq 0$, a contradiction to $x^a y^b \in B_d$. Therefore $B_d = B_{d+1}^-$.

We use this to show $dim_{R/M}(B_d/B_{d-1}) = dim_{R/M} \frac{(M)^{d-1}}{(M)^d}$. It suffices to give a bijection between $\{x^a y^b + M^d \in \frac{M^{d-1}}{M^d} \mid a + b = d - 1\}$ and $\{x^{\bar{a}} y^{\bar{b}} + B_{d-1} \in B_d/B_{d-1} \mid \bar{a} + \bar{b} = -(d + 1)\}$ since these are bases for the corresponding R/M-vector spaces. (Observe that $\{x^{\ell_1} y^{\ell_2} \mid \ell_1 + \ell_2 \geq -(d + 1)\}$ is a basis for B_d and $\{x^{\ell_1} y^{\ell_2} \mid \ell_1 + \ell_2 \geq -(d)\}$ is a basis for B_{d-1}.) The bijection is given by $x^a y^b + M^d \mapsto x^{-a-1} y^{-b-1} + B_d$ noting that $a + b = d - 1$ implies $-a - 1 - b - 1 = -(a + b) - 2 = -(d + 1)$ with inverse map given by $x^{\bar{a}} y^{\bar{b}} + B_d \mapsto x^{-\bar{a}-1} y^{-\bar{b}-1} + M^d$.

In a similar manner, for $R = K[x_1, \ldots, x_n]$ and

$$V = \frac{\langle x_1^{\ell_1} \ldots x_n^{\ell_n} \mid \ell_i \in \mathbb{Z}\rangle}{\langle x_1^{\ell_1} \ldots x_n^{\ell_n} \mid \ell_i \geq 0 \text{ for some } i\rangle}$$

we find that V is a uniform injective module, i.e., $V = E(R/M)$ for the maximal ideal $M = \langle x_1, \ldots, x_n\rangle$ of R. The module V is uniform since each nonzero submodule contains $x_1^{-1} \ldots x_n^{-1}$. Also, $\sqrt{Ann_R(x_1^{-1} \ldots x_n^{-1})} = M$ so $V \subseteq E(R/M)$. In the above case, $n = 2$, we found $B_d = B_{d+1}^- = B_{d+2-1}^-$. For the general case one shows $B_d = B_{d+n-1}^-$. Thus we have a straightforward method for constructing uniform injective modules and hence injective hulls. We summarize in the following theorem.

Theorem 3.3. *Let K be a field, let $R = K[x_1, \ldots, x_n]$, let $W_1 = \langle x_1^{\ell_1} \ldots x_n^{\ell_n} \mid \ell_i \in \mathbb{Z}\rangle \subseteq K(x_1, \ldots, x_n)$, $W_2 = \langle x_1^{\ell_1} \ldots x_n^{\ell_n} \mid$ some $\ell_i \geq 0\rangle \subseteq K(x_1, \ldots, x_n)$, generated as R-modules* *and let* $V = W_1/W_2$. *Then* $V = E(R/M)$ *where* M *is the maximal ideal of* R *generated by* $\{x_1, \dots, x_n\}$.

Example 3.4. We use the above to give an example of an injective R-module W such that $M_R(W)$ is a ring, $M_R(W) = End_R(W)$ and $M_R(W)$ is not commutative. Let $R = K[x,y]$ and V as defined in Theorem 3.3. Now let $W = V \oplus V$ and note from Theorem 3.3 that W is injective. We show W is locally cyclic, i.e., given any two elements $x, y \in W$, there exists $a \in W$ such that $x, y \in Ra$. From this, $M_R(W) = End_R(W)$ [4, Prop. 2.1] and $M_R(W)$ is noncommutative.

Let $(a,b), (c,d) \in W^*$. We find $(e,f) \in W$ and $r, s \in R$ such that $r(e,f) = (a,b)$ and $s(e,f) = (c,d)$. First note that there exists a positive integer N such that $a = \sum_{-N \le i,j < 0}(\alpha_{ij}x^i y^j) + W_2$, $b = \sum_{-N \le i,j<0}(\beta_{ij}x^i y^j) + W_2$, $c = \sum_{-N \le i,j<0}(\gamma_{ij}x^i y^j) + W_2$ and $d = \sum_{-N \le i,j<0}(\delta_{ij}x^i y^j) + W_2$, $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in K$. Let $e = \sum_{-N \le i,j<0}(\alpha_{ij}x^{i-N}y^j + \gamma_{ij}x^i y^{j-N}) + W_2$ and $f = \sum_{-N \le i,j<0}(\beta_{ij}x^{i-N}y^j + \delta_{ij}x^i y^{j-N}) + W_2$. Then $x^N(e,f) = (a,b)$ and $y^N(e,f) = (c,d)$ as desired.

When V is a cyclic module, we know $M_R(V) = End_R(V)$ [7] and since R is commutative one gets $M_R(V)$ is a commutative ring. If D is a Dedekind domain and V is locally cyclic then $M_R(V)$ is a ring. Hence by Theorem 2.11, if V is injective and locally cyclic and R is a Dedekind domain, then $M_R(V)$ is a commutative ring. However, in general as we see in the above example, if R is a commutative Noetherian ring and V is an injective, locally cyclic R-module, $M_R(V)$ need not be commutative.

Example 3.5. More injective hulls over polynomial rings. As above, let K be a field and $R = K[x_1, x_2, \dots, x_n]$. As usual we denote the injective hull of ${}_AV$ by $E_A(V)$ where $V \in$ mod-A. Fix $p \in \{1, 2, \dots, n\}$ and let $F = K(x_{p+1}, \dots, x_n)$ and $S = F[x_1, \dots, x_p]$. Let I and I^e be the ideals generated by $x_1, \dots x_p$ in R and S respectively. Since we have a workable description of $E_S(S/I^e)$ (i.e.,

$$E_S(S/I^e) \cong_S \frac{\langle x_1^{\ell_1} \dots x_p^{\ell_p} \mid \ell_i \in \mathbb{Z}\rangle}{\langle x_1^{\ell_1} \dots x_p^{\ell_p} \mid \text{ some } \ell_i \ge 0\rangle}$$

where $\langle x_1^{\ell_1} \dots x_p^{\ell_p} \mid \ell_i \in \mathbb{Z}\rangle$ and $\langle x_1^{\ell_1} \dots x_p^{\ell_p} \mid \text{some } \ell_i \ge 0\rangle$ are generated as F-vector spaces or S-modules), the same will be true of $E_R(R/I)$ when we show $E_R(R/I) \cong_R E_S(S/I^e)$. To this end, we first show $R_I = S_{I^e}$. Let $a \in S_{I^e}$, $a = \alpha/\beta$ where $\alpha, \beta \in S$, $\beta \notin I^e$. There exists $\gamma \in K[x_{p+1}, \dots, x_n]$ such that $\gamma\alpha$, $\gamma\beta \in R$ and $\gamma\beta \notin I$. Hence $a = \alpha/\beta = \gamma\alpha/\gamma\beta \in R_I$. Since the reverse inclusion is clear the result follows. Further, $I^e S_{I^e} = IS_{I^e} = IR_I$. Therefore $E_S(S/I^e) \cong_{S_{I^e}} E_{S_{I^e}}(S_{I^e}/I^e S_{I^e}) = E_{R_I}(R_I/IR_I) \cong_{R_I} E_{R_I}(R/I)$ and so $E_R(R/I) \cong_R E_S(S/I^e)$.

If $R = K[x_1, \dots, x_n]$ and V is an injective R-module we know $V = \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$ where the P_λ are prime ideals of R. If these P_λ have a rather nice form we can characterize when $M_R(V) = End_R(V)$. We note that the module W of Example 3.4 satisfies our conditions.

Theorem 3.6. *Let $R = K[x_1,\dots,x_n]$, $V = \bigoplus_{\lambda\in\Lambda}(R/P_\lambda)$, where every P_λ is generated by some subset of $\{x_1,\dots,x_n\}$. Then $M_R(V) = End_R(V)$ if and only if for any $x_i \in \{x_1,\dots,x_n\}$, there is at most one P_λ such that $P_\lambda = \langle x_i\rangle$.*

Proof. If $P_{\lambda_1} = \langle x_1\rangle = P_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$ then $M_R(V)$ is not a ring by Theorem 2.14 so the condition is necessary. We establish the converse in a sequence of steps

(1) $E(R/P_\lambda)$ is locally cyclic if P_λ is generated by at least two elements from $\{x_1,\dots,x_n\}$. To see this, suppose $P_\lambda = \langle x_1,\dots,x_k\rangle$, $k \geq 2$ and let

$$0 \neq a, \quad 0 \neq b \in E(R/P_\lambda) \cong \frac{\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \ell_i \in \mathbb{Z}\rangle}{\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \text{ some } \ell_i \geq 0\rangle}$$

where $\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \ell_i \in \mathbb{Z}\rangle$ and $\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \text{some } \ell_i \geq 0\rangle$ are generated as F-vector spaces, $F = K(x_{k+1},\dots,x_n)$. Thus there exists $N \in \mathbb{Z}$, $a = \sum_{N\leq\ell_i<0}\alpha_{\ell_1\dots\ell_k}x_1^{\ell_1}\dots x_k^{\ell_k}$, $\alpha_{\ell_1\dots\ell_k} \in F$ (we are using here only a representative of the coset) and $b = \sum_{N\leq\ell_i<0}\beta_{\ell_1\dots\ell_k}\, x_1^{\ell_1}\dots x_k^{\ell_k}$, $\beta_{\ell_1\dots\ell_k} \in F$. If we take $c = \sum_{N\leq\ell_i<0}(\alpha_{\ell_1\dots\ell_k}x_1^{\ell_1-N}x_2^{\ell_2}\dots x_k^{\ell_k} + \sum_{N\leq\ell_i<0}\beta_{\ell_1\dots\ell_k}\, x_1^{\ell_1}x_2^{\ell_2-N}x_3^{\ell_3}\dots \ell_k^{\ell_k})$ then $x_1^N c = a$ and $x_2^N c = b$.

(2) $E(R/P_\lambda)$ is locally cyclic if $P_\lambda = \langle x_i\rangle$ for some $x_i \in \{x_1,\dots,x_n\}$, where without loss of generality we take $i = 1$. As above let

$$0 \neq a, \quad 0 \neq b \in E(R/P_1) \cong \frac{\langle x_1^{\ell} \mid \ell \in \mathbb{Z}\rangle}{\langle x_1^{\ell} \mid \ell \geq 0\rangle},$$

where $\langle x_1^\ell \mid \ell \in \mathbb{Z}\rangle$ and $\langle x_1^\ell \mid \ell \geq 0\rangle$ are generated as $F = K(x_2,\dots,x_n)$-vector spaces. Then there exist $N_1, N_2 \in \mathbb{Z}$ such that

$$a = \sum_{N_1\leq i<0}\frac{e_i(x_2,\dots,x_n)}{f_i(x_2,\dots,x_n)}x_1^i,$$

$e_{N_1}(x_2,\dots,x_n) \neq 0$ and

$$b = \sum_{N_2\leq i<0}\frac{g_i(x_2,\dots,x_n)}{h_i(x_2,\dots,x_n)}x_1^i,$$

$g_{N_2}(x_2,\dots,x_n) \neq 0$ where we take $N_1 \leq N_2$. Considering $\sum_{N_1\leq i<0}\frac{e_i(x_2,\dots,x_n)}{f_i(x_2,\dots,x_n)}x_1^{i-N_1}$ as an element of $F[[x_1]]$, the power series ring over F, we find there exists $\alpha \in F[[x_1]]$ such that $\alpha(\sum_{N_1\leq i<0}\frac{e_i(x_2,\dots,x_n)}{f_i(x_2,\dots,x_n)}x_1^{i-N_1}) = 1$, i.e., $\sum_{N_1\leq i<0}\frac{e_i(x_2,\dots,x_n)}{f_i(x_2,\dots,x_n)}x_1^{i-N_1} = \alpha^{-1}$ and $\alpha^{-1}x_1^{N_1} = a$.

Let $\beta \in (K[x_1,\dots,x_n])^*$ such that $\beta\alpha^{-1} \in R = K[x_1,\dots,x_n]$. ($\alpha^{-1}$ has only a finite number of nonzero terms.) Let $c = \frac{1}{\beta h_{-1}h_{-2}\dots h_{N_2}}x^{N_1}$. Then for $r = (\beta\alpha^{-1})h_{-1}h_{-2}\dots h_{N_2} \in R$ and $s = \sum_{N_2\leq i<0}\beta g_i h_i' x_1^{i-N_1} \in R$, where $h_i' = \prod_{i\neq j}h_j$, we have $rc = a$ and $sc = \sum_{N_2\leq i<0}\frac{g_i}{h_i}x_1^i = b$.

(3) $F \in M_R(V) \Rightarrow F(a) = \sum_\lambda \varepsilon_\lambda\pi_\lambda(a)$ where π_λ is the projection on the λth component and ε_λ is the insertion into the λth position. We prove this for the case in which $\Lambda = \{1,2,\dots,m\}$. For each $\lambda \in \Lambda$, let Q_λ be the subset of Λ that generates P_λ. Let $a = (a_1,\dots,a_m) \in V$ and $F \in M_R(V)$. Choose a positive integer N such that

$x_i \in Q_j$ implies $x_i^N a_j = 0$, for all i,j. Without loss of generality we assume $Q_1 = \{x_1,\dots,x_k\}$. Then for

$$v \in \frac{\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \ell_i \in \mathbb{Z}\rangle}{\langle x_1^{\ell_1}\dots x_k^{\ell_k} \mid \text{ some } \ell_i \geq 0\rangle}$$

where again these are taken as $F = K(x_{k+1},\dots,x_n)$-vector spaces, we define

$$x_i^M v = \begin{cases} \sum_{L\leq \ell_i<0} (x_i^M \alpha_{\ell_1}\dots\alpha_{\ell_k})x_1^{\ell_1}\dots x_k^{\ell_k}, & \text{if } x_i \notin Q_1, \\ \sum_{L\leq \ell_i<0} (\alpha_{\ell_1}\dots\alpha_{\ell_k})x_1^{\ell_1}\dots x_i^{\ell_i+M}x_{i+1}^{\ell_i+1}\dots x_k^{\ell_k}, & \text{if } x_i \in Q_1, \end{cases}$$

where $M \in \mathbb{Z}$, $\alpha_{\ell_1}\dots\alpha_{\ell_k} \in F$. Inductively extend this definition to all monomials, hence we have extended the action of R on $E(R/P_1)$ to monomials with (possibly) negative exponents.

Define a similar action for the other $E(R/P_\lambda)$. Now let $\alpha = \prod_{i\geq 2} x_i^{-N}$ and $\beta = x_1^{-N}$ (hence $\alpha^{-1} = \prod_{i\geq 2} x_i^N$ and $\beta^{-1} = x_1^N$). Then $\beta^{-1}(\alpha a_1, \beta a_2,\dots,\beta a_m) = (0,a_2,\dots,a_m)$ and $\alpha^{-1}(\alpha a_1,\beta a_2,\dots,\beta a_m) = (a_1,0,\dots,0)$. Then $f(\alpha^{-1}+\beta^{-1})(\alpha a_1,\beta a_2,\dots,\beta a_m) = f[(a_1,0,\dots,0)+(0,a_2,\dots,a_m)]$ while $(\alpha^{-1}+\beta^{-1})f(\alpha a_1,\beta a_2,\beta a_m) = f(a_1,0,\dots,0)+f(0,a_2,\dots,a_m)$. Continuing we obtain $f((a_1,\dots,a_m)) = f((a_1,0,\dots,0)) + f((0,a_2,0,\dots,0)) + \dots + f((0,\dots,0,a_m))$ as desired.

Combining (1), (2) and (3) now gives $M_R(V) = End_R(V)$. □

References

[1] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra (Addison-Wesley, Reading, MA, 1969).

[2] P. Fuchs and C.J. Maxson, Centralizer near-rings determined by PID-modules, Arch. Math. 56 (1991) 140–147.

[3] P. Fuchs, C.J. Maxson and G. Pilz, On rings for which homogeneous maps are linear, Proc. Amer. Math. Soc. 112 (1991) 1–7.

[4] J. Hansen and J.A. Johnson, Centralizer near-rings that are rings, J. Austral. Math. Soc., to appear.

[5] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math. 8 (1958) 511–528.

[6] C.J. Maxson and J.D.P. Meldrum, Centralizer representations of near-fields, J. Algebra 89 (1984) 73–89.

[7] C.J. Maxson and K.C. Smith, Centralizer near-rings determined by local rings, Houston J. Math. 11 (1985) 355–366.

[8] C.J. Maxson and A.P.J. van der Walt, Centralizer near-rings over free ring modules, J. Austral. Math. Soc. 50 (1991) 279–291.

[9] D.S. Passman, A Course in Ring Theory (Wadsworth and Brooks/Cole, Pacific Grove, CA, 1991).

[10] G. Pilz, Near-Rings (North-Holland/American Elsevier, Amsterdam, 1983).

[11] R.Y. Sharp, Steps in commutative algebra, London Math. Soc. Student Texts, Vol. 19 (Cambridge University Press, Cambridge, 1990).

[12] D.W. Sharpe and P. Valmos, Injective modules, Cambridge Tracts in Mathematics, Vol. 62 (Cambridge University Press, Cambridge, 1972).

[13] A.B. van der Merwe, Near-rings of homogeneous maps, to appear.

[14] R. Ware and J. Zelmanowitz, Simple endomorphism rings, Amer. Math. Monthly 77 (1970) 987–989.